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Convergence of Bézier Triangular Nets and a Theorem of Pólya

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This paper is concerned with Bernstein–Bézier triangular patches and their Bézier nets. By degree raising, a sequence of Bézier nets is obtained. It is known that the sequence converges uniformly to the Bernstein–Bézier triangular patch determined by those nets. A new proof of the convergence, which is more geometric and constructive, is presented. Connections of convergence and a theorem due to Pólya are revealed. Extensions to higher dimensional cases are also mentioned.

1. INTRODUCTION

Let T be a given triangle. Each point P in T has barycentric coordinates (u, v, w) with respect to T. The triple (u, v, w) satisfies the conditions

$$\begin{gathered} u \geqslant 0, \qquad v \geqslant 0, \qquad w \geqslant 0, \\ u+v+w=1. \end{gathered} \tag{1}$$

We identify P and its barycentric coordinates by writing $P=(u, v, w)$. Let n be any positive integer.

The subdivision of T into n^2 congruent triangles with vertices at $(i/n, j/n, k/n)$, in which $i+j+k=n$, denoted by $S_n(T)$, is called the nth subdivision of T. The points $(i/n, j/n, k/n)$, $i+j+k=n$, are called nodes of $S_n(T)$. $S_4(T)$ is illustrated in Fig. 1.

Given is a set f of $(n+1)(n+2)/2$ real numbers, i.e., $f := \{f_{i,j,k} \mid i+j+k=n\}$, and the polynomial

$$B^n(f; p) := \sum_{i+j+k=n} f_{i,j,k} \frac{n!}{i!\, j!\, k!} u^i v^j w^k \tag{2}$$

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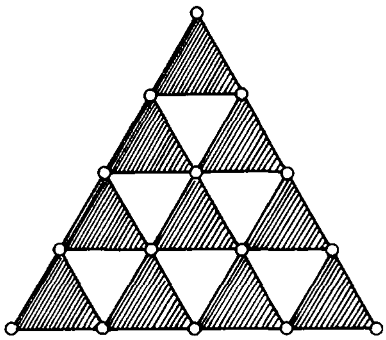

FIG. 1. $S_4(T)$ with its nodes.

is defined as the Bernstein–Bézier (B-B) polynomial of f over the triangle T. $f_{i,j,k}(i+j+k=n)$ are called the Bézier ordinates of $B^n(f;p)$ while $(i/n, j/n, k/n; f_{i,j,k})$ are called its Bézier points. The point set $(P; B^n(f;p))$ with $p \in T$ forms a surface patch over triangle T. We simply call polynomial (2) the B-B triangular patch with domain triangle T. The piecewise linear function $\hat{f}(p)$ which is linear on each subtriangle of $S_n(T)$ and interpolates to $f_{i,j,k}$ at $(i/n, j/n, k/n)$, is said to be the Bézier net of patch (2). Figure 2 illustrates a Bézier net and the corresponding patch (with $n=3$).

It is known [3] that if we set

$$Ef_{i,j,k} := \frac{1}{n+1}(if_{i-1,j,k} + jf_{i,j-1,k} + kf_{i,j,k-1}), \tag{3}$$

where $i+j+k=n+1$, and write

$$Ef := \{Ef_{i,j,k} \mid i+j+k=n\},$$

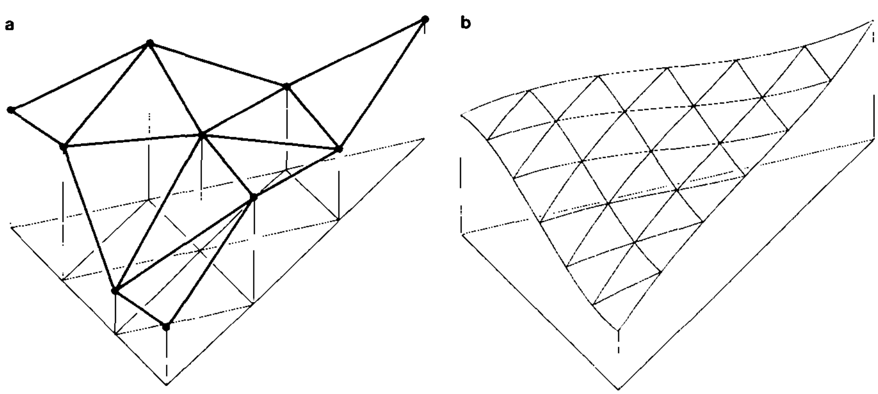

FIG. 2. Bézier net and the corresponding patch ($n=3$).

then we have

$$B^n(f; p) = B^{n+1}(Ef; p).$$

This means that it is always possible to write $B^n(f; p)$ as a B-B polynomial of degree $n+1$. The technique just mentioned is called degree raising. The Bézier net associated with Ef is denoted by $E\hat{f}(p)$ which is linear on each subtriangle of $S_{n+1}(T)$ and interpolates $Ef_{i,j,k}$ at $(i/(n+1), j/(n+1), k/(n+1))$.

If one repeats the process of degree raising, a sequence of Bézier nets $\hat{f}(p)$, $E\hat{f}(p)$, $E^2\hat{f}(p)$, ..., will be obtained. It has been proved that

THEOREM 1 [3]. *We have*

$$\lim_{m \to \infty} E^m\hat{f}(p) = B^n(f; p) \tag{4}$$

uniformly on T.

Recently we found that Theorem 1 has a very close connection with a famous theorem, which appeared in the early stage of this century [6], in the algebraic theory of polynomials in several variables. For historical remarks, see [5]. To present Pólya's theorem we need some definitions. A real *form* is a homogeneous polynomial $F(x_1, x_2, ..., x_m)$, with real coefficients, in m variables. A form is said to be strictly positive, in a certain region of the variables, if $F > 0$ for all points in that region.

THEOREM 2 (Pólya). *If the form $F(x_1, x_2, ..., x_m)$ is strictly positive in the region*

$$(x_1, x_2, ..., x_m) \qquad x_1 \geqslant 0, \qquad x_2 \geqslant 0, ..., x_m \geqslant 0$$

and

$$x_1 + x_2 + \cdots + x_m > 0,$$

then F may be expressed as

$$F = \frac{G}{H}, \tag{5}$$

where G and H are forms with positive coefficients. In particular, we may suppose that

$$H = (x_1 + x_2 + \cdots + x_m)^p$$

for a suitable natural number p.

In the present paper, we first show that Theorem 2 can be derived from Theorem 1, and then point out that Pólya's technique for the proof of his theorem, with further modifications, in turn provides a proof for Theorem 1 which is more geometric and constructive than existing ones.

2. Proof of Theorem

For simplicity of writing we suppose $m=3$. No new point of principle arises for general m.

A form F in three variables u, v, w can be expressed by

$$F(u, v, w)=\sum_{\alpha+\beta+\gamma=n} f_{\alpha,\beta,\gamma} \frac{n!}{\alpha!\,\beta!\,\gamma!} u^{\alpha}v^{\beta}w^{\gamma}, \tag{6}$$

in which u, v, w are independent. If $F>0$ in the region $u \geqslant 0$, $v \geqslant 0$, $w \geqslant 0$ and $u+v+w>0$, then F has a positive minimum, say τ, in the region $u \geqslant 0$, $v \geqslant 0$, $w \geqslant 0$ and $u+v+w=1$. In this case (6) becomes a B-B polynomial on the triangle T.

An elementary manipulation brings the following identity

$$(u+v+w)^m F=\frac{m!\,n!}{(m+n)!} \sum_{a+b+c=m+n} \sum_{i+j+k=n} f_{i,j,k} \times \binom{a}{i}\binom{b}{j}\binom{c}{k} \frac{(a+b+c)!}{a!\,b!\,c!} u^a v^b w^c. \tag{7}$$

For a proof, see [5, 8]. If $(u, v, w) \in T$, i.e., $u+v+w=1$, then (7) can be viewed as the mth degree raising of the B-B polynomial $B^n(f; p)$. Hence we have

$$E^m f_{i,j,k}=\frac{m!\,n!}{(m+n)!} \sum_{\alpha+\beta+\gamma=n} f_{\alpha,\beta,\gamma} \binom{i}{\alpha}\binom{j}{\beta}\binom{k}{\gamma}, \tag{8}$$

in which $i+j+k=m+n$.

By Farin's theorem, the inequality

$$E^m \hat{f}(p) \geqslant \frac{\tau}{2} > 0$$

holds for all P in T and for sufficiently large m. Particularly,

$$E^m f_{i,j,k}=E^m \hat{f}\left(\frac{i}{m+n}, \frac{j}{m+n}, \frac{k}{m+n}\right) > 0 \tag{9}$$

for $i+j+k=m+n$. We denote the form in the right-hand side of (7) by G_m. Equation (9) shows that all coefficients of G_m are positive for sufficiently large m. Identity (7) gives

$$F(u, v, w) = \frac{G_m(u, v, w)}{(u+v+w)^m}$$

which is the desired representation for sufficiently large m.

The strict positivity of B-B polynomials was characterized by Zhou [8]. Obviously he was not aware of Theorem 2.

3. An Alternate Proof of Theorem 1

There are several proofs for the theorem. The original proof [3] is very short but some sophisticated results by Stancu are involved. The proof given by Zhou ([8]; see also [4]) is relatively elementary but it does not provide a proof for the *uniform* convergence. For other proofs the reader is referred to [1, 7] in which the structure of Bézier nets has been carefully studied.

We define for real x and nonnegative i the usual binomial coefficient $\binom{x}{i}$ as

$$\binom{x}{0} = 1,$$

$$\binom{x}{i} = \frac{x(x-1)\cdots(x-i+1)}{i!}, \qquad i = 1, 2, 3, \ldots.$$

Consider the following polynomial of degree n:

$$L_n(f; p) = \sum_{\alpha+\beta+\gamma=n} f_{\alpha,\beta,\gamma} \binom{nu}{\alpha}\binom{nv}{\beta}\binom{nw}{\gamma}. \tag{10}$$

It is easy to verify that

$$L_n\left(f; \frac{i}{n}, \frac{j}{n}, \frac{k}{n}\right) = f_{i,j,k}, \qquad i+j+k=n.$$

This means that (10) is the Lagrange interpolation to $\hat{f}(p)$ at all nodes of $S_n(T)$.

In particular, if $f_{\alpha,\beta,\gamma} = 1$ for all $\alpha+\beta+\gamma=n$, from (10) we have the following identity,

$$\sum_{\alpha+\beta+\gamma=n} \binom{nu}{\alpha}\binom{nv}{\beta}\binom{nw}{\gamma} = 1,$$

for $u+v+w=1$ and $n=1, 2, 3, \ldots$. In general, we have

$$\sum_{\alpha+\beta+\gamma=n} \binom{a}{\alpha}\binom{b}{\beta}\binom{c}{\gamma} = \binom{a+b+c}{n}. \tag{11}$$

The Lagrange interpolation to $E^m \hat{f}(p)$ at all nodes of $S_{m+n}(T)$, by (10) and (8), is

$$\frac{m!\,n!}{(m+n)!} \sum_{i+j+k=n} f_{i,j,k} \sum_{\alpha+\beta+\gamma=m+n} \binom{\alpha}{i}\binom{\beta}{j}\binom{\gamma}{k}\binom{A}{\alpha}\binom{B}{\beta}\binom{C}{\gamma}, \tag{12}$$

in which $A := (m+n)u$, $B := (m+n)v$, $C := (m+n)w$. It is clear that

$$\binom{\alpha}{i}\binom{A}{\alpha} = \binom{A}{i}\binom{A-i}{\alpha-i}, \qquad \binom{\beta}{j}\binom{B}{\beta} = \binom{B}{j}\binom{B-j}{\beta-j},$$

$$\binom{\gamma}{k}\binom{C}{\gamma} = \binom{C}{k}\binom{C-k}{\gamma-k},$$

and by (11) that

$$\sum_{\alpha+\beta+\gamma=m+n} \binom{A-i}{\alpha-i}\binom{B-j}{\beta-j}\binom{C-k}{\gamma-k} = 1,$$

as

$$A-i+B-j+C-k = (A+B+C)-(i+j+k) = m+n-n = m$$

and

$$\alpha-i+\beta-j+\gamma-k = m+n-n = m.$$

Hence (12) becomes

$$\frac{m!\,n!}{(m+n)!} \sum_{i+j+k=n} f_{i,j,k} \binom{(m+n)u}{i}\binom{(m+n)v}{j}\binom{(m+n)w}{k}. \tag{13}$$

Define

$$\varphi(u, v, w; t) := n!\, t^n \sum_{i+j+k=n} f_{i,j,k} \binom{ut^{-1}}{i}\binom{vt^{-1}}{j}\binom{wt^{-1}}{k}, \qquad 0 \leqslant t \leqslant 1. \tag{14}$$

We can verify that (13) is equal to

$$\lambda_m \varphi\left(u, v, w; \frac{1}{m+n}\right), \tag{15}$$

where

$$\lambda_m := \frac{(m+n)^n}{(m+1)(m+2)\cdots(m+n)}, \tag{16}$$

$m = 1, 2, 3, ...$ It is obvious that $\lim_{t \to 0} \varphi(p; t) = F(p)$, as

$$t^i \binom{ut^{-1}}{i} = \frac{u(u-t)(u-2t)\cdots(u-(i-1)t)}{i!} \to \frac{u^i}{i!} \qquad (t \to 0),$$

etc. If we define $\varphi(p; 0) := F(p)$, then $\varphi(p; t)$ is continuous on the region

$$u \geqslant 0, \qquad v \geqslant 0, \qquad w \geqslant 0, u+v+w=1, \qquad 0 \leqslant t \leqslant 1. \tag{17}$$

Function $\varphi(p; t)$ represents a family of surfaces with a single parameter $t \in [0, 1]$. Especially we have mentioned that the surface patch $\lambda_m \varphi(p; 1/(m+n))$ coincides with the Bézier net $E^m \hat{f}(p)$ at all its vertices.

The investigation of convergence of $E^m \hat{f}(p)$ is now shifted to that of $\lambda_m \varphi(p; 1/(m+n))$. The second problem is easier than the first as $\varphi(p; t)$ has an analytical expression on T, while $E^m \hat{f}(p)$, being a piecewise linear function, does not. By the mean value theorem of univariate functions we know that

$$\varphi(p; t') - \varphi(p; t) = O(|t' - t|), \qquad t' \to t$$

in the region (17). In particular we have

$$\varphi\left(p; \frac{1}{m+n}\right) - F(p) = \varphi\left(p; \frac{1}{m+n}\right) - \varphi(p; 0) = O\left(\frac{1}{m}\right), \qquad \text{as} \quad m \to \infty.$$

Since $\lambda_m = 1 + O(1/m)$, we still have

$$\lambda_m \varphi\left(p; \frac{1}{m+n}\right) - F(p) = O\left(\frac{1}{m}\right). \tag{18}$$

We have shown that the sequence of surfaces $\lambda_m \varphi(p; 1/(m+n))$ converges uniformly to the B-B patch $B^n(f; p)$ with the rate $O(1/m)$ as $m \to \infty$. Now we have to estimate the difference between $\lambda_m \varphi(p; 1/(m+n))$ and the corresponding Bézier net $E^m \hat{f}(p)$.

Take a typical upward subtriangle with vertices

$$\left(\frac{i+1}{m+n}, \frac{j}{m+n}, \frac{k}{m+n}\right),$$

$$\left(\frac{i}{m+n}, \frac{j+1}{m+n}, \frac{k}{m+n}\right), \qquad \left(\frac{i}{m+n}, \frac{j}{m+n}, \frac{k+1}{m+n}\right),$$

in which $i+j+k=m+n-1$ (see Fig. 3).

Let P be any point inside the subtriangle and P has the barycentric coordinates (λ, μ, ν) with respect to the subtriangle. Hence the barycentric coordinates (u, v, w) with respect to the domain triangle T will be

$$\left(\frac{i+\lambda}{m+n}, \frac{j+\mu}{m+n}, \frac{k+\nu}{m+n}\right).$$

Being linear on the subtriangle, $E^m\hat{f}(p)$ is a linear convex combination of its values at three vertices of the subtriangle; more precisely,

$$E^m\hat{f}(p) = \lambda E^m f_{i+1,j,k} + \mu E^m f_{i,j+1,k} + \nu E^m f_{i,j,k+1}$$

which becomes by (8)

$$E^m\hat{f}(p) = \frac{n!\,m!}{(n+m)!} \sum_{\alpha+\beta+\gamma=n} f_{\alpha,\beta,\gamma} \left[\lambda \binom{i+1}{\alpha}\binom{j}{\beta}\binom{k}{\gamma} + \mu \binom{i}{\alpha}\binom{j+1}{\beta}\binom{k}{\gamma} + \nu \binom{i}{\alpha}\binom{j}{\beta}\binom{k+1}{\gamma}\right]. \qquad (19)$$

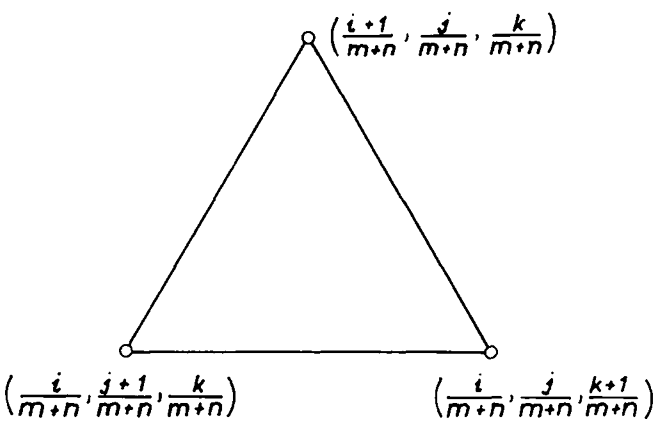

FIG. 3. A typical upward subtriangle.

On the other hand, by (14) we have

$$\lambda_m \varphi\left(p; \frac{1}{m+n}\right) = \lambda_m \varphi\left(\frac{i+\lambda}{m+n}, \frac{j+\mu}{m+n}, \frac{k+\nu}{m+n}; \frac{1}{m+n}\right)$$
$$= \frac{m!\,n!}{(m+n)!} \sum_{\alpha+\beta+\gamma=n} f_{\alpha,\beta,\gamma} \binom{i+\lambda}{\alpha}\binom{j+\mu}{\beta}\binom{k+\nu}{\gamma}. \quad (20)$$

Define

$$\psi(\lambda, \mu, \nu) := \binom{i+\lambda}{\alpha}\binom{j+\mu}{\beta}\binom{k+\nu}{\gamma}.$$

By Taylor expansion we obtain

$$\psi(\lambda, \mu, \nu) - \psi(1, 0, 0) = (\lambda - 1)\frac{\partial\psi}{\partial\lambda}(\lambda^*, \mu^*, \nu^*)$$
$$+ \mu \frac{\partial\mu}{\partial\mu}(\lambda^*, \mu^*, \nu^*) + \nu \frac{\partial\psi}{\partial\nu}(\lambda^*, \mu^*, \nu^*),$$

where $(\lambda^*, \mu^*, \nu^*)$ is some point in T. It is clear that

$$\left|\binom{i+\lambda}{\alpha}\right| \leqslant \binom{i+\alpha}{\alpha}, \qquad \left|\binom{j+\mu}{\beta}\right| \leqslant \binom{j+\beta}{\beta}, \qquad \left|\binom{k+\nu}{\gamma}\right| \leqslant \binom{k+\gamma}{\gamma}.$$

Since

$$\frac{\partial}{\partial\lambda}\binom{i+\lambda}{\alpha} = \frac{1}{\alpha!}\sum_{\tau=0}^{\alpha-1} \frac{(\lambda+i)(\lambda+i-1)\cdots(\lambda+i-\alpha+1)}{\lambda+i-\tau},$$

similar estimation shows that

$$\left|\frac{\partial}{\partial\lambda}\binom{i+\lambda}{\alpha}\right| \leqslant \binom{\alpha+i}{\alpha-1}$$

and then

$$\left|\frac{\partial\psi}{\partial\lambda}\right| = \left|\frac{\partial}{\partial\lambda}\binom{i+\alpha}{\alpha}\right| \left|\binom{j+\mu}{\beta}\right| \left|\binom{k+\nu}{\gamma}\right| \leqslant \binom{i+\alpha}{\alpha-1}\binom{j+\beta}{\beta}\binom{k+\gamma}{\gamma}.$$

Similar inequalities hold for $|\partial\psi/\partial\mu|$ and $|\partial\psi/\partial\nu|$. Therefore

$$\left|\binom{i+\lambda}{\alpha}\binom{j+\mu}{\beta}\binom{k+\nu}{\gamma}-\binom{i+1}{\alpha}\binom{j}{\beta}\binom{k}{\gamma}\right|$$
$$=|\psi(\lambda,\mu,\nu)-\psi(1,0,0)|$$
$$\leqslant\left|\frac{\partial\psi}{\partial\lambda}\right|+\left|\frac{\partial\psi}{\partial\mu}\right|+\left|\frac{\partial\psi}{\partial\nu}\right|$$
$$\leqslant\binom{i+\alpha}{\alpha-1}\binom{j+\beta}{\beta}\binom{k+\gamma}{\gamma}$$
$$+\binom{i+\alpha}{\alpha}\binom{j+\beta}{\beta-1}\binom{k+\gamma}{\gamma}+\binom{i+\alpha}{\alpha}\binom{j+\beta}{\beta}\binom{k+\gamma}{\gamma-1}.$$

Hence by (10)

$$\left|\sum_{\alpha+\beta+\gamma=n} f_{\alpha,\beta,\gamma}\left[\binom{i+\lambda}{\alpha}\binom{j+\mu}{\beta}\binom{k+\nu}{\gamma}-\binom{i+1}{\alpha}\binom{j}{\beta}\binom{k}{\gamma}\right]\right|$$
$$\leqslant 3\,\|f\|\binom{m+2n-1}{n-1}$$

in which

$$\|f\|:=\max\{|f_{\alpha,\beta,\gamma}|:\alpha+\beta+\gamma=n\}$$

and (11) has been used. Finally

$$\left|\lambda_m\varphi\left(p;\frac{1}{m+n}\right)-E^m\hat{f}(p)\right|\leqslant\frac{m!\,n!}{(m+n)!}\,3\,\|f\|\binom{m+2n-1}{n-1}=O\left(\frac{1}{m}\right). \tag{21}$$

The same estimate is valid for P lying on the downward subtriangles of $S_{m+n}(T)$. Combining (18) and (21) we get

$$F(p)-E^m\hat{f}(p)=O\left(\frac{1}{m}\right)\qquad(m\to\infty). \tag{22}$$

This completes the proof of Theorem 1.

4. Higher Dimension Cases

For $P\in\mathbb{R}^s$ and any $s+1$ affinely independent points $T_i\in\mathbb{R}^s$, $i=0,1,\ldots,s$, there are $s+1$ real numbers $\lambda_0,\lambda_1,\ldots,\lambda_s$ uniquely determined by de Boor [1]

$$P=\sum_{i=0}^{s}\lambda_i T_i$$

and

$$\sum_{i=0}^{s} \lambda_i = 1.$$

$(\lambda_0, \lambda_1, ..., \lambda_s)$ are called the barycentric coordinates of P with respect to the s-simplex T spanned by T_i, $i = 0, 1, ..., s$. We write $\lambda = (\lambda_0, \lambda_1, ..., \lambda_s)$. For a set i of $s+1$ nonnegative integers $i_0, i_1, ..., i_s$, we define

$$i := (i_0, i_1, ..., i_s),$$
$$|i| := i_0 + i_1 + \cdots + i_s,$$
$$i! = i_0!\, i_1! \cdots i_s!,$$
$$\lambda^i := \lambda_0^{i_0} \lambda_1^{i_1} \cdots \lambda_s^{i_s}.$$

With any set of scalars $f := \{f_i \mid |i| = n\}$, we define the Bernstein–Bézier polynomial on the simplex T by

$$B^n(f; \lambda) := \sum_{|i| = n} f_i \frac{n!}{i!} \lambda^i, \tag{23}$$

in which f is called the set of Bézier ordinates for $B^n(f; \lambda)$. The degree raising technique is the same. It is easy to show that

$$E^m f_j = \frac{m!\, n!}{(m+n)!} \sum_{|i| = n} f_i \binom{j}{i}, \tag{24}$$

where

$$\binom{j}{i} := \binom{j_0}{i_0} \binom{j_1}{i_1} \cdots \binom{j_s}{i_s},$$

and $|j| = m + n$. Formula (24) generalizes (8). For $s > 2$, it has been rightfully stressed and detailed by Dahmen and Micchelli [2] that, since there are several equally reasonable subdivisions of the simplex T, the Bézier nets $\hat{f}(\lambda)$ could not be uniquely determined. Similar to (14), we define

$$\varphi(\lambda; t) := n!\, t^n \sum_{|i| = n} f_i \binom{\lambda t^{-1}}{i} \tag{25}$$

for $t > 0$, where

$$\binom{\lambda t^{-1}}{i} := \binom{\lambda_0 t^{-1}}{i_0} \binom{\lambda_1 t^{-1}}{i_1} \cdots \binom{\lambda_s t^{-1}}{i_s}.$$

We can show easily that the function

$$\frac{(m+n)^n}{(m+1)(m+2)\cdots(m+n)}\,\varphi\left(\lambda;\frac{1}{m+n}\right)$$

interpolates to $E^m f_i$ at $i/(m+n)$ in which $|i|=m+n$ and that

$$\frac{(m+n)^n}{(m+1)(m+2)\cdots(m+n)}\,\varphi\left(\lambda;\frac{1}{m+n}\right)-B^n(f;\lambda)=O\left(\frac{1}{m}\right).$$

In a word, the results in previous sections of our paper can be extended in an obvious manner. Therefore we arrive at the extension of (20),

$$B^n(f;\lambda)-E^m\hat{f}(\lambda)=O\left(\frac{1}{m}\right),$$

in which $E^m\hat{f}(\lambda)$ denotes any reasonable piecewise linear interpolant to the data points

$$\left(\frac{i}{m+n};E^m f_i\right),\qquad |i|=n+m.$$

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